Linear programming

Chapter V: Duality Theory

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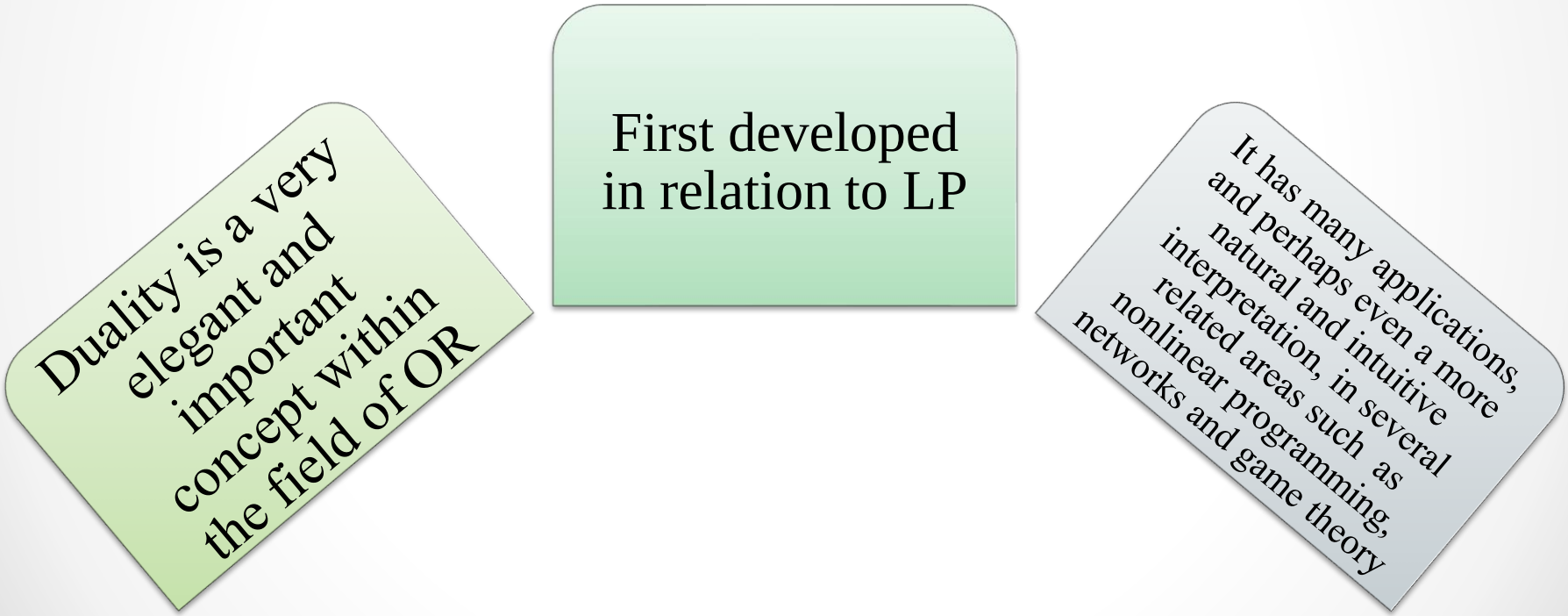
Introduction & Motivation


First developed in relation to LP
Duality is a very elegant and important concept within the field of OR
It has many applications, and perhaps even a more natural and intuitive interpretation, in several related areas such as nonlinear programming, networks and game theory

Introduction & Motivation

The notion of duality within LP asserts that every LP has associated with it a related LP called its **dual**

The original problem in relation to its dual is termed the **primal**

It is the **relationship** between the primal and its dual, both on a mathematical and economic level, that is truly the essence of duality theory

Example 1

There is a small company in Melbourne which has recently become engaged in the production of office furniture.

The company manufactures tables, desks and chairs.

The production of a table requires 8 Kgs of wood and 5 Kgs of metal and is sold for $80.

Example 1-continued

A desk uses 6 Kgs of wood and 4 Kgs of metal and is sold for $60; and a chair requires 4 Kgs of both metal and wood and is sold for $50.

We would like to determine the revenue maximizing strategy for this company, given that their resources are limited to 100 Kgs of wood and 60 Kgs of metal.

Formulation-example 1

$$\max_{x} \; Z = 80x_1 + 60x_2 + 50x_3$$

$$8x_1 + 6x_2 + 4x_3 \leq 100$$

$$5x_1 + 4x_2 + 4x_3 \leq 60$$

$$x_1, x_2, x_3 \geq 0$$

Duality-example 1

Now consider that there is a much bigger company in Melbourne which has been the only producer of this type of furniture for many years.

They don't appreciate the competition from this new company; so they have decided to tender an offer to buy all of their competitor's resources and therefore put them out of business.

Formulating the dual LP

The challenge for this large company then is to develop a linear program which will determine the appropriate amount of money that should be offered for a unit of each type of resource, such that the offer will be acceptable to the smaller company while minimizing the expenditures of the larger company.

Problem D1

$$\min_{y} \; w = 100y_1 + 60y_2$$

$$8y_1 + 5y_2 \geq 80$$

$$6y_1 + 4y_2 \geq 60$$

$$4y_1 + 4y_2 \geq 50$$

$$y_1, y_2 \geq 0$$

Example 2: A Diet Problem

An individual has a choice of two types of food to eat, meat and potatoes, each offering varying degrees of nutritional benefit.

He has been warned by his doctor that he must receive at least 400 units of protein, 200 units of carbohydrates and 100 units of fat from his daily diet

Example 2-continued

Given that a kg of steak costs $10 and provides 80 units of protein, 20 units of carbohydrates and 30 units of fat, and that a kg of potatoes costs $2 and provides 40 units of protein, 50 units of carbohydrates and 20 units of fat, he would like to find the minimum cost diet which satisfies his nutritional requirements.

Formulating example 2

$$\min_{x} Z = 10x_1 + 2x_2$$

$$80x_1 + 40x_2 \geq 400$$

$$20x_1 + 50x_2 \geq 200$$

$$30x_1 + 20x_2 \geq 100$$

$$x_1, x_2 \geq 0$$

The Dual of example 2

Now consider a chemical company which hopes to attract this individual away from his present diet by offering him synthetic nutrients in the form of pills.

This company would like to determine prices per unit for their synthetic nutrients which will bring them the highest possible revenue while still providing an acceptable dietary alternative to the individual.

Dual Problem of example 2

$$\max_{y} \; w = 400y_1 + 200y_2 + 100y_3$$

$$80y_1 + 20y_2 + 30y_3 \le 10$$

$$40y_1 + 50y_2 + 20y_3 \le 2$$

$$y_1, y_2, y_3 \ge 0$$

Comments

Each of the two examples describes some kind of **competition** between two decision makers.

The notion of "competition" could be investigated more formally in "**Game Theory**".

We shall investigate the economic interpretation of the primal/dual relationship later in this chapter.

Definition

Primal Problem

$$z^* := \max_x \; Z = cx$$

$$s.t.$$

$$Ax \le b$$

$$x \ge 0$$

Dual Problem

$$w^* := \min_x \; w = yb$$

$$s.t.$$

$$yA \ge c$$

$$y \ge 0$$

b is not assumed to be non-negative

Example

$$Max\ Z = 10X_1 + 5X_2$$

$$2X_1 + 3X_2 \leq 10$$

$$6X_1 + X_2 \leq 15$$

$$4X_1 - 5X_2 \leq 35$$

$$X_1, X_2 \geq 0$$

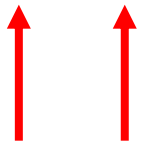
$$\textit{Max } Z_X = 10X_1 + 5X_2$$

$$2X_1 + 3X_2 \leq 10 \longleftarrow Y_1$$

$$6X_1 + X_2 \leq 15 \longleftarrow Y_2$$

$$4X_1 - 5X_2 \leq 35 \longleftarrow Y_3$$

$$X_1, X_2 \geq 0$$



1st constraint 2nd constraint

$$\textit{Min } Z_Y = Y_1^+ \quad Y_2^+ \quad Y_3$$

$$Y_1^+ \quad Y_2^+ \quad Y_3 \geq$$

$$Y_1 \quad Y_2 \quad Y_3 \geq$$

$$Y_1, Y_2, Y_3 \geq 0$$

Example

Primal

$$\max_{x} \; Z = 5x_1 + 3x_2 - 8x_3 + 0x_4 + 12x_5$$

$$3x_1 - 8x_2 + 9x_4 - 15x_5 \le 20$$

$$18x_1 + 5x_2 - 8x_3 + 4x_4 + 12x_5 \le 30$$

$$x_1, x_2, x_3, x_4, x_5 \ge 0$$

Dual

$$\min_{y} w = 20y_1 + 30y_2$$

$$\begin{aligned}
3y_1 + 18y_2 &\geq 5 \\
-8y_1 + 5y_2 &\geq 3 \\
-8y_2 &\geq -8 \\
9y_1 + 4y_2 &\geq 0 \\
-15y_1 + 12y_2 &\geq 12 \\
y_1, y_2 &\geq 0
\end{aligned}$$

Primal-Dual relationship

		$x_1 \geq 0$	$x_2 \geq 0$		$x_n \geq 0$		w =
	$y_1 \geq 0$	a_{11}	a_{12}		a_{1n}	$\leq$	b_1
Dual	$y_2 \geq 0$	a_{21}	a_{22}		a_{2n}	$\leq$	b_2
(min w)	…	…	…			…	…
	$y_m \geq 0$	a_{m1}	a_{m2}		a_{mn}	$\leq$	b_n
		$\geq$	$\geq$		$\geq$		
	z=	c_1	c_2		c_n		

Example

$$\max_{x} \; Z = 4x_1 + 10x_2 - 9x_3$$

$$\begin{aligned}
5x_1 - 18x_2 + 5x_3 &\le 15 \\
-8x_1 + 12x_2 - 8x_3 &\le 8 \\
12x_1 - 4x_2 + 8x_3 &\le 10 \\
2x_1 - 5x_3 &\le 5 \\
x_1, x_2, x_3 &\ge 0
\end{aligned}$$

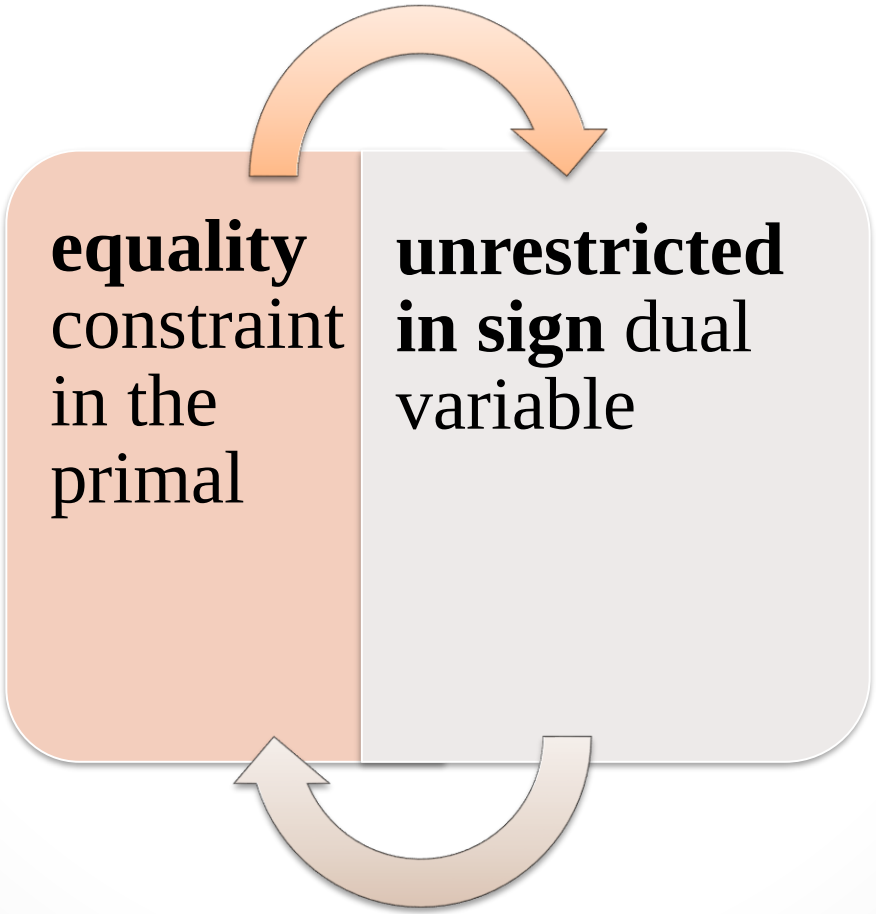
Dual

$$\min_{y} w = 15y_1 + 8y_2 + 10y_3 + 5y_4$$

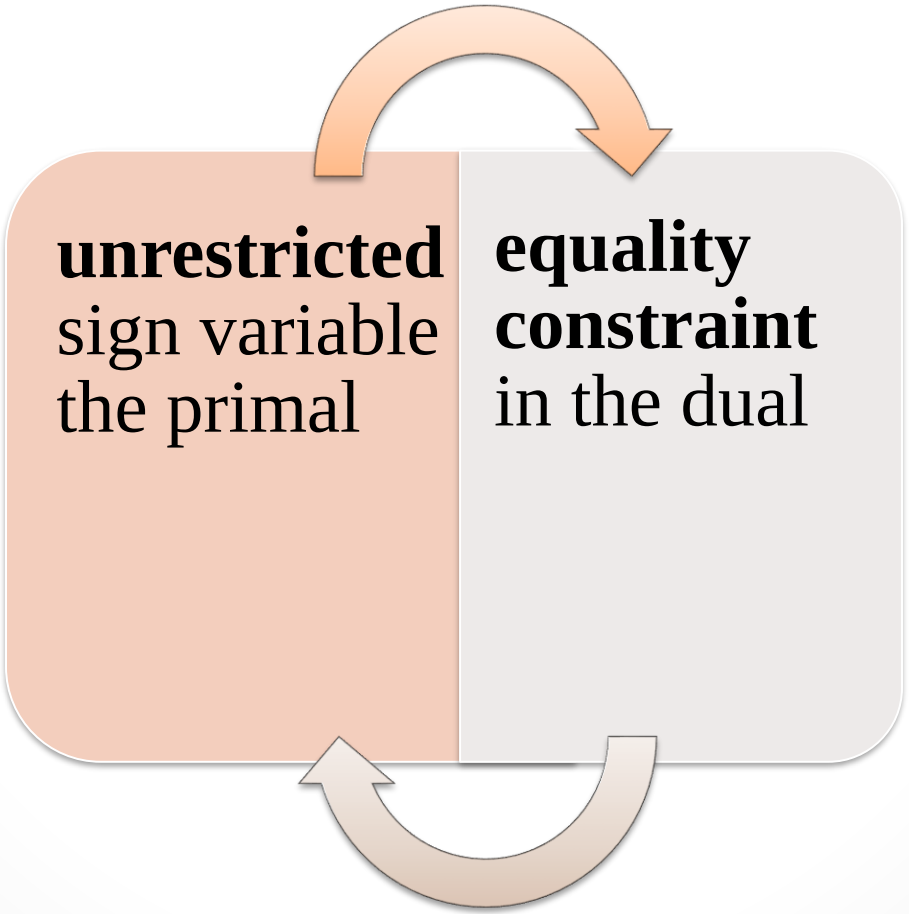
$$\begin{aligned}
5y_1 - 8y_2 + 12y_3 + 2y_4 &\geq 4 \\
-18y_1 + 12y_2 - 4y_3 &\geq 10 \\
5y_1 + 8y_3 - 5y_4 &\geq -9 \\
y_1, y_2, y_3, y_4 &\geq 0
\end{aligned}$$

FINDING THE DUAL OF NON-CANONICAL LP

Streamlining the conversion ...



Streamlining the conversion ...



Primal-Dual relationship

Primal Problem	Dual Problem
opt=max	opt=min
Constraint i : <= form = form	Variable i : y_i >= 0 y_i urs
Variable j: x_j >= 0 x_j urs	Constraint j: >= form = form

Example

$$\max_{x} Z = 5x_1 + 4x_2$$

$$3x_1 - 8x_2 \geq -6$$

$$x_1 + 6x_2 = -5$$

$$8x_1 = 10$$

$$x_2 \geq 0; \quad x_1 \text{ urs}$$

Equivalent non-standard form

$$\max_{x} Z = 5x_1 + 4x_2$$

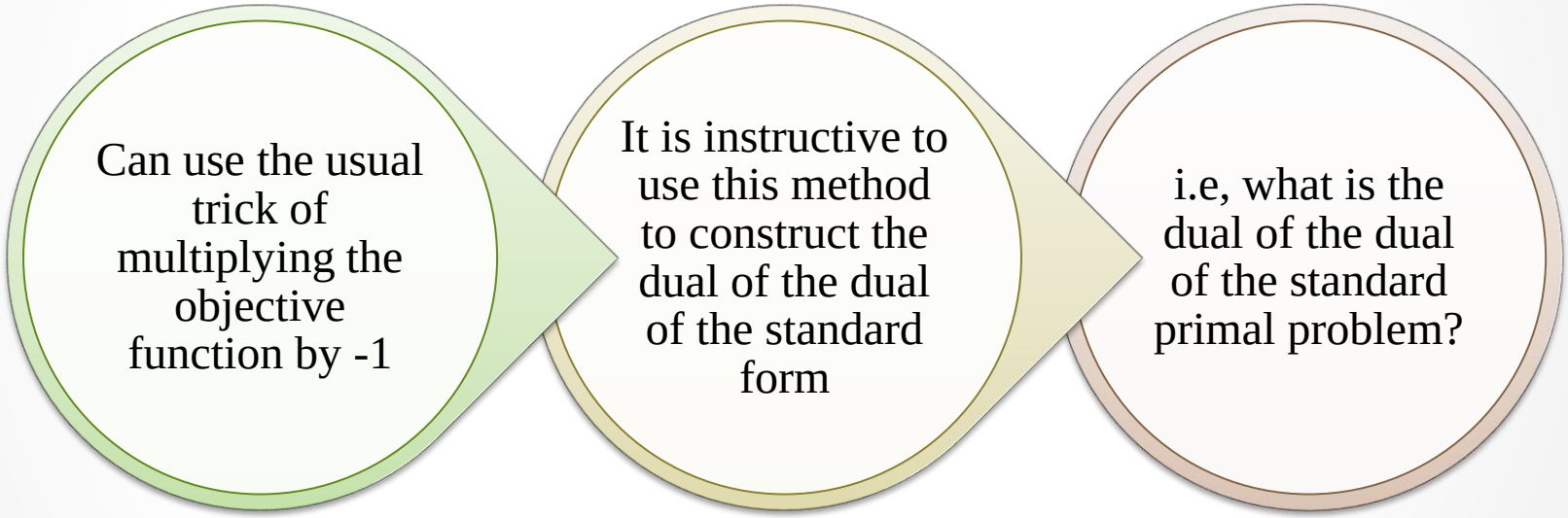
$$\begin{aligned} -3x_1 + 8x_2 &\le 6 \\ x_1 + 6x_2 &= -5 \\ 8x_1 &= 10 \\ x_2 \ge 0;\ \ x_1 &\text{ urs} \end{aligned}$$

Dual from the recipe

$$\min_{y} w = 6y_1 - 5y_2 + 10y_3$$

$$\begin{aligned} -3y_1 + \ \ y_2 + 8y_3 &= 5 \\ 8y_1 + 6y_2 \qquad\quad &\geq 4 \\ y_1 \geq 0;\ \ y_2, y_3 \ &\text{urs} \end{aligned}$$

What about min ?


Can use the usual trick of multiplying the objective function by -1
It is instructive to use this method to construct the dual of the dual of the standard form
i.e, what is the dual of the dual of the standard primal problem?

What is the dual of

$$w^* := \min_x \; w = yb$$

$$s.t.$$

$$yA \geq c$$

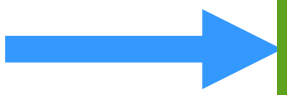
$$y \geq 0$$

$$\max_{y} -w = -yb$$

$$s.t.$$

$$yA \geq c$$

$$y \geq 0$$



$$\max_{y} -w = -yb$$

$$s.t.$$

$$-yA \leq -c$$

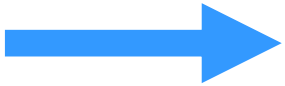
$$y \geq 0$$

$$\min_{x} -Z = -cx$$

$$s.t.$$

$$-Ax \geq -b$$

$$x \geq 0$$



$$\max_{x} Z = cx$$

$$s.t.$$

$$Ax \leq b$$

$$x \geq 0$$

Important Observation

FOR ANY PRIMAL LINEAR PROGRAM, THE DUAL OF THE DUAL IS THE PRIMAL

Primal-Dual Relationship

Primal or Dual	Dual or Primal
opt=max	opt=min
Constraint i : <= form = form	Variable i : y_i >= 0 y_i urs
Variable j: x_j >= 0 x_j urs	Constraint j: >= form = form

Example

$$\min_{x} \; Z = 6x_1 + 4x_2$$

$$3x_1 - 5x_2 \geq 12$$

$$x_1 + 2x_2 = -8$$

$$5x_1 - \quad x_2 \leq 10$$

$$x_1, x_1 \geq 0$$

Equivalent form

$$\min_{x} \; Z = 6x_1 + 4x_2$$

$$3x_1 - 5x_2 \geq 12$$

$$x_1 + 2x_2 = -8$$

$$-5x_1 + x_2 \geq -10$$

$$x_1, x_2 \geq 0$$

Dual

$$\max_{y} \; w = 12y_1 - 8y_2 - 10y_3$$

$$\begin{aligned} 3y_1 + \; y_2 - 5y_3 &\le 6 \\ -5y_1 + 2y_2 + \; y_3 &\le 4 \\ y_1, y_3 \ge 0; \;\; y_2 &\text{ urs} \end{aligned}$$

Maximization	Minimization
Number of constraints	Number of variables
constraint ≤	Variable positive or zero
constraint ≥	Variable negative or zero
constraint =	Unconstrained Variable
Number of variables	Number of constraints
Variable positive or zero	constraint ≥
Variable negative or zero	constraint ≤
Unconstrained Variable	constraint =
Coefficient of the j^{th} variable In the objective function	RHS of the j^{th} constraint
RHS of the i^{th} constraint	Coefficient of the i^{th} variable In the objective function
Technological coefficient of the j^{th} variable in the i^{th} constraint	Technological coefficient of the i^{th} variable in the j^{th} constraint

Example

Primal	Dual
$\text{Max } Z_X = 5X_1 + 4X_2 - 2X_3$	$\text{Min } Z_Y = 20Y_1 + 5Y_2 + 30Y_3$
$2X_1 + 3X_2 \leq 20$	$2Y_1 + Y_2 + Y_3 \geq 5$
$X_1 - 4X_3 \geq 5$	$3Y_1 + Y_3 \geq 4$
$X_1 + X_2 + X_3 = 30$	$-4Y_2 + Y_3 = -2$
$X_1, X_2 \geq 0$	$Y_1 \geq 0, Y_2 \leq 0, Y_3 \in IR$

Primal	Dual
$\text{Min } ZX = 15X_1 + 2X_2$	$\text{Max } Z_Y = 10Y_1+50Y_2+2Y_3+120Y_4$
$X_1 - X_2 \geq 10$	$Y_1 + 7Y_2 - Y_3 + 9Y_4 \leq 15$
$7X_1 + 2X_2 \leq 50$	$-Y_1 + 2Y_2 +3Y_3 + Y_4 = 2$
$-X_1 + 3X_2 \geq 2$	$Y_1 \geq 0, Y_2 \leq 0, Y_3 \geq 0, Y_4 \in IR$
$9X_1 + X_2 = 120$	
$X_1 \geq 0, X_2 \in IR$	

Note that the **dual of the dual** is the **primal**

Results

- Consider the primal and dual LP:

$$\max Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{Primal LP } s.t. \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i, i = 1, m$$

$$x_j \ge 0, j = 1, n$$

$$\min W = \sum_{i=1}^{m} b_i y_i$$

$$\text{Dual LP } s.t. \quad \sum_{i=1}^{m} a_{ji} y_i \ge c_j, j = 1, n$$

$$y_i \ge 0, i = 1, m$$

Theorem

Theorem. The dual of the dual is the primal.

Proof: The dual problem LP2 is equivalent to the following problem:

$$\begin{aligned} \text{Max} \quad & -b^T y \\ \text{s. t.} \quad & -A^T y \le -c \\ & y \ge 0. \end{aligned}$$

By definition, the dual of this problem is

$$\begin{aligned} \text{Min} \quad & -c^T x \\ \text{s. t.} \quad & (-A^T x \ge -b \\ & x \ge 0, \end{aligned}$$

or equivalently,

$$\begin{aligned} \text{Max} \quad & c^T x \\ \text{s. t.} \quad & Ax \le b \\ & x \ge 0. \end{aligned}$$

which is the primal problem.

Various theorems related to duality (without proof)

If x is primal feasible and y is dual feasible, then $c^T x \leq b^T y$ (weak duality theorem).

If x* is primal feasible and y* is dual feasible and $c^T x^* = b^T y^*$ then x* and y* are optimal

If one of a pair of primal and dual problems has an optimal solution, then the other also has an optimal solution and the optimal values of their objective functions are equal.

Theorem-continued

If the primal is feasible and unbounded then the dual is infeasible.

If the dual is feasible and unbounded, then the primal is infeasible.

Complementary Slackness Theorem: A primal feasible solution x^* and a dual feasible solution y^* are optimal if and only if $(b-Ax^*)^T y^* = 0$ and $(A^T y^* - c)^T x^* = 0$.

Interpretation of Duality

The dual variable y^* measures the change in optimal profit due to a unit change in resource i.

The fact that profit would increase by y_i^* if an additional unit of resource i were available imputes a value or price to resource i.

This value or price is called a shadow price. Thus, the shadow price is the amount of contribution of an additional unit of a resource to total profit.

Discussion

- Note from the weak duality theorem that we always have

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i$$

Discussion-continued

The inequality is strict except at optimal solution where equality holds

At optimality, y_i is interpreted as the worth of resource i

Weak duality theorem:

the profit is ≤ worth of resources

Resources are not exploited according to the best allocation except at optimality

y_i's: shadow prices of resources

Discussion-continued

Shadow price: additional unit profit that could be made by acquiring additional units of resource i

Resource not totally consumed at optimal exploitation:

- related constraint is inactive
- corresponding shadow price must be zero
- corresponding slack variable is basic

Active constraint (i.e., the resource is entirely consumed at optimal exploitation):

- corresponding shadow price is positive
- it will be profitable to acquire additional units of that resource.

Example

Consider the following LP and its dual :

$$\textit{Max}\ Z_X = 5X_1 + 2X_2 + 7X_3$$

$$X_1 + X_2 + X_3 \leq 50$$

$$2X_1 + 4X_2 + 3X_3 \leq 75$$

$$X_1, X_2, X_3 \geq 0$$

$$\textit{Min}\ Z_Y = 50Y_1 + 75Y_2$$

$$Y_1 + 2Y_2 \geq 5$$

$$Y_1 + 4Y_2 \geq 2$$

$$Y_1 + 3Y_2 \geq 7$$

$$Y_1 \geq 0\,,\ Y_2 \geq 0$$

The optimal tableau is given by

C_J		5	2	7	0	0	
	Basis	X_1	X_2	X_3	S_1	S_2	b_I
0	S_1	0	-1	-1/2	1	-1/2	12,5
5	X_1	1	2	3/2	0	1/2	37,5
	Δ_J	0	-8	-1/2	0	-5/2	187,5
Dual		E_1	E_2	E_3	Y_1	Y_2	Z_Y
Solution		0	8	1/2	0	5/2	187,5

Economic Interpretation of the dual problem

Assume that in the context of the previous LP, 3 products P_1, P_2 et P_3 are to be manufactured

The production process requires the use of 2 resources R_1 & R_2 , which are available in limited quantities of 50 et 75 units respectively

Reconsider the following example

$$Max\ Z_X = 5X_1 + 2X_2 + 7X_3$$

$$X_1 + X_2 + X_3 \leq 50 \qquad (R_1)$$

$$2X_1 + 4X_2 + 3X_3 \leq 75 \qquad (R_2)$$

$$X_1, X_2, X_3 \geq 0$$

C_J		5	2	7	0	0	
	Basis	X_1	X_2	X_3	S_1	S_2	b_i
0	S_1	0	-1	-1/2	1	-1/2	12,5
5	X_1	1	2	3/2	0	1/2	37,5
	C_J-Z_J	0	-8	-1/2	0	-5/2	187,5

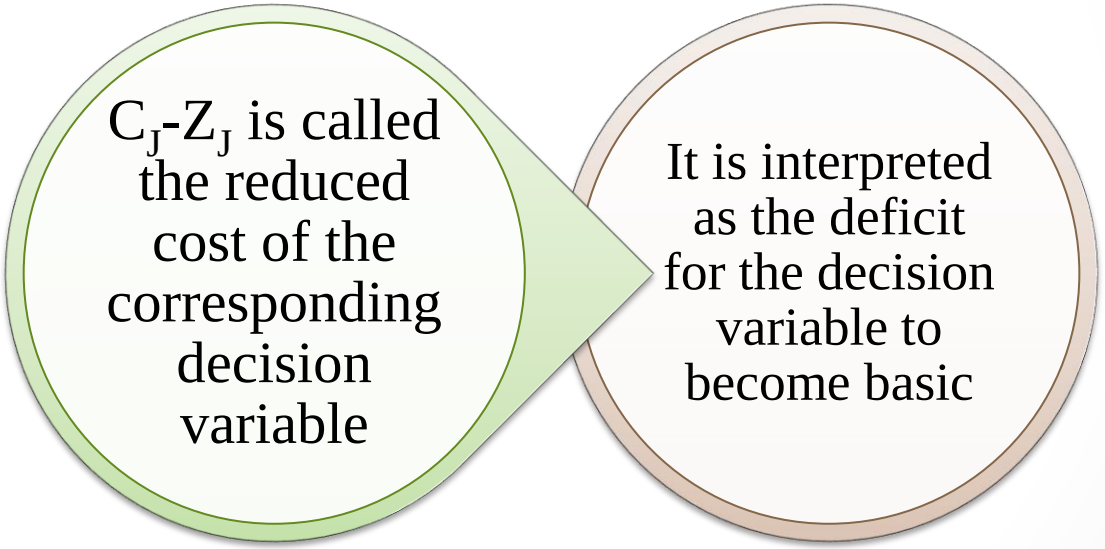
From the corresponding optimal Tableau, the marginal value or shadow price of R_1 is 0 and the shadow price of R_2 is 2.5

An additional unit of R_1 has no effect on the value of Z, whereas an additional unit of R_2 will increase Z by 2.5 dinars

The producer will not be ready to cede a unit of R_1 (respectively R_2) unless he obtains against it at least 0 (respectively 2.5 dinars) additional gain

Equivalently, in order to acquire one more unit of R_1 (respectively R_2), the producer will be willing to pay additional cost of 0 (respectively 2.5 dinars)

The C_J-Z_J of a decision variable


C_J-Z_J is called the reduced cost of the corresponding decision variable
It is interpreted as the deficit for the decision variable to become basic

The C_J-Z_J of a decision variable

In the last tableau, C_2-Z_2 =-8. This says that the unit profit of P_2 must increase at least by 8 dinars for P_2 to become profitable.

By forcing one unit of P_2 to be produced, the profit will be reduced by 8 dinars.

The unit profit of P_3 must increase at least by 0.5 dinars for P_3 to become profitable.